

A COUNTABLY COMPACT TOPOLOGICAL GROUP H SUCH THAT $H \times H$ IS NOT COUNTABLY COMPACT

KLAAS PIETER HART AND JAN van MILL

ABSTRACT. Using $\mathbf{MA}_{\text{countable}}$ we construct a topological group with the properties mentioned in the title.

0. INTRODUCTION

In this paper we construct, assuming Martin's Axiom for countable posets ($\mathbf{MA}_{\text{countable}}$), a countably compact topological group H for which H^2 is not countably compact. The existence of such a group under $\mathbf{MA}$ was announced by van Douwen in [vD] but was never published.

Our construction is like van Douwen's construction in [vD] in that our group is a subgroup of c2 and that we construct the points of our group in an induction of length $\mathfrak{c}$. But whereas van Douwen used $\mathbf{MA}$ to construct a countably compact subgroup of c2 without convergent sequences and then constructed two countably compact subgroups of this group with a countable intersection, we construct our group all at once. Moreover, our group H can be written as $G + D$ with D a countable subgroup and G an ω -bounded subgroup, so that our group has many convergent sequences. Also van Douwen needed $\mathbf{MA}$ for certain uncountable posets, we get by using $\mathbf{MA}_{\text{countable}}$ only.

The interest in examples such as the ones in the present paper comes from the fact that by Comfort and Ross [CR] the product of an arbitrary family of pseudocompact topological groups is pseudocompact. (For more information see [vD].)

This paper is organized as follows: §1 contains some definitions and preliminaries. In §2 we use our technique to construct two countably compact subgroups H_0 and H_1 of c2 such that $H_0 \times H_1$ is not countably compact, and in §3 we construct our main example H . The reason for doing this is that the construction in §3 is rather involved notationwise so that it may be useful to have a more transparent version available. Also, in §3, we essentially construct a countably compact subgroup H of ${}^c2 \times {}^c2$ which contains (modified

Received by the editors June 6, 1988 and, in revised form, February 8, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54B10, 54D30, 54G20, 22A99, 03E50.

Key words and phrases. Countably compact topological group, ω -independent family, Martin's Axiom for countable posets.

versions of) $H_0 \times \{0\}$ and $\{0\} \times H_1$ so that H^2 contains $H_0 \times H_1$ is a closed noncountably compact subgroup.

1. DEFINITIONS AND PRELIMINARIES

1.0. Notation. We follow the usual conventions regarding ordinals and cardinals. As always $\mathfrak{c} = 2^\omega$. For any set X we let

$$[X]^{\leq \omega} = \{A \subseteq X : |A| \leq \omega\}, \quad [X]^{< \omega} = \{A \subseteq X : |A| < \omega\},$$

$$[X]^\omega = [X]^{\leq \omega} \setminus [X]^{< \omega}.$$

The ordinal $2 = \{0, 1\}$ is a topological group with $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$, and when given the discrete topology. If X is any set then ${}^X 2$ (the set of functions from X to 2) is a compact topological group when given the product group structure and the product topology. Its zero-element, the function $X \times \{0\}$, is written $\underline{0}$. If A is a subset of ${}^X 2$ then $\langle\langle A \rangle\rangle$ denotes the subgroup generated by A . Note that $\langle\langle A \rangle\rangle$ is finite when A is finite (this follows from the fact that ${}^X 2$ is Boolean, i.e., $\forall x: x + x = \underline{0}$).

1.1. Independent families and ω -bounded subgroups. In our construction we shall need an ω -independent family of size $\mathfrak{c}$ on $\mathfrak{c}$. This is a family $\mathcal{A}$ of subsets of $\mathfrak{c}$ satisfying: if $\mathcal{A}', \mathcal{A}'' \in [\mathcal{A}]^{\leq \omega}$ are disjoint then $|\bigcap \mathcal{A}' \setminus \mathcal{A}''| = \mathfrak{c}$. Such families exist; an easy example can be obtained as follows: for $\alpha \in \mathfrak{c}$ let

$$A_\alpha = \{F \in [\mathfrak{c} \times \mathfrak{c}]^{\leq \omega} : F \cap (\{\alpha\} \times \mathfrak{c}) \neq \emptyset\}.$$

Then $\mathcal{A} = \{A_\alpha : \alpha \in \mathfrak{c}\} \subseteq [\mathfrak{c} \times \mathfrak{c}]^{\leq \omega}$ satisfies the stronger condition

$$\text{if } K \in [\mathfrak{c}]^{\leq \omega} \text{ then } \left| \bigcap_{\alpha \in K} A_\alpha \setminus \bigcup_{\alpha \in \mathfrak{c} \setminus K} A_\alpha \right| = \mathfrak{c}.$$

Of course $\mathcal{A}$ lives not on $\mathfrak{c}$ but on $[\mathfrak{c} \times \mathfrak{c}]^{\leq \omega}$, a set of size $\mathfrak{c}$.

(a) Let $\mathcal{B}$ be an ω -independent family on a set X of size $\mathfrak{c}$. We identify $\mathcal{B}$ with its set of characteristic functions. Thus, we can consider $G = \langle\langle \mathcal{B} \rangle\rangle$ the group generated by $\mathcal{B}$. We then let

$$G^+ = \bigcup \{\bar{E} : E \in [G]^{\leq \omega}\}$$

$$= \bigcup \{\bar{E} : E \in [G]^{\leq \omega} \text{ and } E \text{ is a subgroup of } G\}.$$

Then G^+ is a subgroup of ${}^X 2$ and it is easy to see that if $E \subseteq G^+$ is countable then $\bar{E} \subseteq G^+$. It follows that G^+ is ω -bounded (this means that if $E \subseteq G^+$ is countable then $\bar{E}$ (now closure in G^+) is compact) and in fact G^+ is the smallest ω -bounded subgroup of ${}^X 2$ containing $\mathcal{B}$. We call G^+ the ω -bounded subgroup generated by $\mathcal{B}$.

(b) In (a) if $x \in G^+ \setminus \{0\}$ then we can find $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2 \in [\mathcal{B}]^{\leq \omega}$ such that $\bigcap \mathcal{B}_0 \setminus \bigcup \mathcal{B}_1 \subseteq x^{-1}(1) \subseteq \bigcup \mathcal{B}_2$, and $\mathcal{B}_0 \cap \mathcal{B}_1 = \emptyset$.

Proof. Let $E \subseteq G$ be countable such that $x \in \overline{E}$. For each $e \in E$ we have a finite set $F_e \subseteq \mathcal{B}$ such that $e = \sum_{B \in F_e} B$. Put $\mathcal{B}_2 = \bigcup_{e \in E} F_e$. Now if $\alpha \notin \bigcup \mathcal{B}_2$ then $e(\alpha) = 0$ for all $e \in E$, as $x \in \overline{E}$ we get $x(\alpha) = 0$. Hence $x^+(1) \subseteq \bigcup \mathcal{B}_2$. Next pick $\alpha \in x^+(1)$. The set $U = \{y \in {}^X 2 : y(\alpha) = 1\}$ is a neighborhood of x in ${}^X 2$, hence $x \in \overline{U \cap E}$. For $e \in U \cap E$ put $F_e^+ = \{B \in F_e : \alpha \in B\}$ and $F_e^- = F_e \setminus F_e^+$. Clearly if $\mathcal{B}_0 = \bigcup \{F_e^+ : e \in U \cap E\}$ and $\mathcal{B}_1 = \bigcup \{F_e^- : e \in U \cap E\}$ then $\alpha \in \bigcap \mathcal{B}_0 \setminus \bigcup \mathcal{B}_1$.

Now let $\gamma \in \bigcap \mathcal{B}_0 \setminus \bigcup \mathcal{B}_1$ and $e \in U \cap E$. Then $B \in F_e^+$ implies $\gamma \in B$ and $B \in F_e^-$ implies $\gamma \notin B$. It follows that $e(\gamma) = e(\alpha) = 1$. So as $x \in \overline{U \cap E}$, also $x(\gamma) = 1$. Hence $\bigcap \mathcal{B}_0 \setminus \bigcup \mathcal{B}_1 \subseteq x^+(1)$. $\square$

(c) Let $\mathcal{B}$ be an ω -independent family on $\mathfrak{c}$. For every $B \in \mathcal{B}$ let X_B be such that $|X_B \Delta B| < \mathfrak{c}$. Then $\{X_B : B \in \mathcal{B}\}$ is also ω -independent.

Proof. If $\mathcal{B}', \mathcal{B}'' \subseteq \mathcal{B}$ are countable and disjoint, pick α such that for all $B \in \mathcal{B}' \cup \mathcal{B}''$, $X_B \Delta B \subseteq \alpha$. Then $(\bigcap_{B \in \mathcal{B}'} X_B \setminus \bigcup_{B \in \mathcal{B}''} X_B) \cap [\alpha, \mathfrak{c}) = (\bigcap \mathcal{B}' \setminus \bigcup \mathcal{B}'') \cap [\alpha, \mathfrak{c})$ has cardinality $\mathfrak{c}$. $\square$

1.2. u -limits. In one case we will need the concept of a u -limit where $u \in \omega^*$ (the set of free ultrafilters on ω). If $\{x_n : n \in \omega\}$ is a sequence in a topological space X and $x \in X$ then we say $x = u\text{-}\lim x_n$ iff $\{n : x_n \in O\} \in u$ for every neighborhood O of x .

We need the following: if X is countably compact and if $\{x_n : n \in \omega\}$ is a sequence in X then for some $u \in \omega^*$, $u\text{-}\lim x_n$ exists. For let $x \in X \setminus \{x_n : n \in \omega\}$ be a point such that for every open neighborhood O of x , $\{n : x_n \in O\}$ is infinite. The set of all such subsets of ω is a free filter $\mathcal{F}$: for any $u \in \omega^*$ extending $\mathcal{F}$, $x = u\text{-}\lim x_n$.

1.3. Martin's Axiom for countable posets ($\text{MA}_{\text{countable}}$). In topological language this statement says that the real line cannot be covered by fewer than $\mathfrak{c}$ nowhere dense sets. The partial order form, which is the form that we shall use, says the following:

if $\mathbf{P}$ is a countable partially ordered set and if $\mathcal{D}$ is a collection of fewer than $\mathfrak{c}$ dense sets in $\mathbf{P}$ then there is a filter G on $\mathbf{P}$ intersecting every element of $\mathcal{D}$.

Here a set $D \subseteq \mathbf{P}$ is *dense* iff $(\forall p \in \mathbf{P})(\exists q \leq p)(q \in D)$ and a set $G \subseteq \mathbf{P}$ is a *filter* iff

$$\forall p \in G \forall q \in \mathbf{P} : p \leq q \rightarrow q \in G \quad \text{and} \\ \forall p, q \in G \exists r \in G : r \leq p, q.$$

In this paper we shall apply $\text{MA}_{\text{countable}}$ to posets which are subsets of posets of the form $\text{Fn}(I, x)$, with I countable and x finite. Here

$$\text{Fn}(I, x) = \{p : p \text{ is a finite function, } \text{dom}(p) \subseteq I, \text{ran}(p) \subseteq x\},$$

ordered by reverse inclusion: $p \leq q$ iff $p \supseteq q$. In this case a set D is dense if every p has an extension q such that $q \in D$, and a set G is a filter if it is closed

under taking restrictions and if for any two elements p and q of G there is an $r \in G$ extending both p and q . If G is a filter then $\bigcup G$ is a function. Usually for every $i \in I$, $D_i = \{p: i \in \text{dom}(p)\}$ is dense. If $G \cap D_i \neq \emptyset$ for every i then $\text{dom}(\bigcup G) = I$. (For more information on Martin's Axiom see [We].)

2. CONSTRUCTION I

As mentioned in the introduction the construction in the next section needs a lot of notation and this might obscure the main idea. We therefore present an easier version giving two countably compact groups H_0 and H_1 with $H_0 \times H_1$ not countably compact. Also the construction §3 amounts to making a countably compact group H containing (versions of) H_0 and H_1 as closed subgroups and such that $H_0 \cap H_1 = \{0\}$.

2.0. What we construct. Using $\text{MA}_{\text{countable}}$ we shall construct three subgroups D , G_0 , and G_1 of c2 satisfying:

- (i) D is countable,
- (ii) G_0 and G_1 are ω -bounded,
- (ii) $G_0 \cap G_1 = \{0\} = D \cap (G_0 + G_1)$, and
- (iv) if $E \subseteq D$ is infinite then E has accumulation points in G_0 and G_1 .

Actually $\text{MA}_{\text{countable}}$ is used only in the construction of D .

2.1. Why this works. If we have D , G_0 , and G_1 as in 2.0 then we let

$$H_0 = D + G_0 \quad \text{and} \quad H_1 = D + G_1.$$

(a) H_0 and H_1 are countably compact.

Proof. Let $i \in \{0, 1\}$ and let $I = \{d_n + g_n: n \in \omega\} \subseteq H_i$ be infinite. If $\{d_n: n \in \omega\}$ is finite then, since G_i is ω -bounded, $\text{Cl}_H I$ is compact. If $\{d_n: n \in \omega\}$ is infinite let $g \in G_i$ be an accumulation point of it. Pick $u \in \omega^*$ such that $g = u\text{-}\lim d_n$. Then $h = u\text{-}\lim g_n \in G_i$ again since G_i is ω -bounded. It follows that $g + h$ is an accumulation point of I in H . $\square$

(b) $H_0 \times H_1$ is not countably compact.

Proof. By (iii) $H_0 \cap H_1 = D$ so that $\{\langle d, d \rangle: d \in D\}$ is a countable and closed subgroup of $H_0 \times H_1$. $\square$

2.2. Explanation. To begin we take our ω -independent family $\mathcal{A}$ on $\mathfrak{c}$ from 1.1, and we split $\mathcal{A}$ into two families $\mathcal{A}_0$ and $\mathcal{A}_1$ of size $\mathfrak{c}$. We enumerate these as $\langle A_\alpha^0: \omega \leq \alpha < \mathfrak{c} \rangle$ and $\langle A_\alpha^1: \omega \leq \alpha < \mathfrak{c} \rangle$ respectively, without repetitions.

As we construct D we shall modify the sets A_α^0 and A_α^1 a bit to obtain sets B_α^0 and B_α^1 respectively such that $|B_\alpha^i \cap A_\alpha^i| < \mathfrak{c}$ for all α and i . We shall let $\mathcal{B}_i = \{B_\alpha^i: \omega \leq \alpha < \mathfrak{c}\}$ ($i \in 2$) and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$. By 1.1(c) $\mathcal{B}$ will still be ω -independent. Finally then G_0 and G_1 will be the ω -bounded subgroups of c2 generated by $\mathcal{B}_0$ and $\mathcal{B}_1$ respectively.

Take $x \in G_0 \setminus \{0\}$ and $y \in G_1 \setminus \{0\}$. By 1.1(b) we find countable $K, L, M \subseteq [\omega, c)$ such that

$$\bigcap_{\alpha \in K} B_\alpha^0 \setminus \bigcup_{\alpha \in L} B_\alpha^0 \subseteq x^\leftarrow(1) \quad \text{and} \quad y^\leftarrow(1) \subseteq \bigcup_{\alpha \in M} B_\alpha^1.$$

But then

$$\bigcap_{\alpha \in K} B_\alpha^0 \setminus \left(\bigcup_{\alpha \in L} B_\alpha^0 \cup \bigcup_{\alpha \in M} B_\alpha^1 \right)$$

has cardinality c and it is contained in $x^\leftarrow(1) \setminus y^\leftarrow(1)$. It follows that $G_0 \cap G_1 = \{0\}$.

Next we consider how to make sure that $D \cap (G_0 + G_1) = \{0\}$. For this observe that $G_0 + G_1$ is in fact the ω -bounded subgroup of ${}^c 2$ generated by $\mathcal{B}$. Now if $x \in (G_0 + G_1) \setminus \{0\}$ then by 1.1(b) there is a countable $\mathcal{B}' \subseteq \mathcal{B}$ with $x^\leftarrow(1) \subseteq \mathcal{B}'$. Thus a sufficient condition for $D \cap (G_0 + G_1) = \{0\}$ would be

$$(\forall d \in D \setminus \{0\})(\forall \mathcal{B}' \in [\mathcal{B}]^\omega)(d^\leftarrow(1) \setminus \bigcup \mathcal{B}' \neq \emptyset).$$

However we know $\mathcal{B}$ only at the end of the construction. But we do know $\mathcal{A}$ in advance and a moments reflection shows that the following is sufficient:

$$(v) \quad (\forall d \in D \setminus \{0\})(\forall \mathcal{A}' \in [\mathcal{A}]^{\leq \omega})(|d^\leftarrow(1) \setminus \bigcup \mathcal{A}'| = c).$$

Finally we describe what D will look like. We put $I = \{x \in {}^\omega 2 : x^\leftarrow(1) \text{ is finite}\}$. We will have $D = \{d_x : x \in I\}$ such that for all $x \in I$, $d_x \upharpoonright \omega = x$ and for all $x, y \in I$, $d_x + d_y = d_{x+y}$. We already have 2.0(i) and we indicated how we shall take care of 2.0(ii) and 2.0(iii). For 2.0(iv) we enumerate $[I]^\omega$ as $\{E_\alpha : \omega \leq \alpha < c\}$. During our construction we will take at step α an accumulation point y_α of $\{d_x \upharpoonright \alpha : x \in E_\alpha\}$ and put $B_\alpha^i = y_\alpha \cup A_\alpha^i \upharpoonright [\alpha, c)$ ($i \in 2$). For the rest of the construction we will make sure that B_α^0 and B_α^1 will be accumulation points of $\{d_x : x \in E_\alpha\}$. We also have to take care of (v). For this we let $\langle \langle x_\alpha, \mathcal{E}_\alpha \rangle : \omega \leq \alpha < c \rangle$ enumerative

$$\{\langle x, \mathcal{E} \rangle : x \in I \setminus \{0\}, \mathcal{E} \in [\mathcal{A}]^{\leq \omega}\}$$

in such a way that for every $\langle x, \mathcal{E} \rangle$ the set $\{\alpha : \langle x_\alpha, \mathcal{E}_\alpha \rangle = \langle x, \mathcal{E} \rangle \text{ and } \alpha \notin \bigcup \mathcal{E}\}$ has cardinality c . Then (v) follows once we have $d_{x_\alpha}(\alpha) = 1$ for all α .

Now that we know what to do, it only remains to do it.

2.3. The construction of D . We construct $D \upharpoonright \alpha = \{d_x \upharpoonright \alpha : x \in I\}$ by induction on $\alpha \in [\omega, c]$. In the end we put $d_x = d_x \upharpoonright c$ and $D = \{d_x : x \in I\}$. At every stage once $D \upharpoonright \alpha$ is found we let $y_\alpha \in {}^\alpha 2$ be an accumulation point of $\{d_x \upharpoonright \alpha : x \in E_\alpha\}$ and we put, as promised,

$$B_\alpha^i = y_\alpha \cup A_\alpha^i \upharpoonright [\alpha, c) \quad (i \in 2).$$

Our inductive hypotheses are

$$(0) \quad \forall x \in I : d_x \upharpoonright \omega = x,$$

- (1) $\forall x, y \in I: d_x \upharpoonright \alpha + d_y \upharpoonright \alpha = d_{x+y} \upharpoonright \alpha$,
- (2) if $\omega \leq \beta < \alpha$ then $d_{x_\beta}(\beta) = 1$ and $\forall x \in I \ d_x \upharpoonright \beta \subseteq d_x \upharpoonright \alpha$, and
- (3) if $\omega \leq \beta < \alpha$ then $B_\beta^0 \upharpoonright \alpha$ and $B_\beta^1 \upharpoonright \alpha$ are accumulation points of $\{d_x \upharpoonright \alpha: x \in E_\beta\}$.

If $\alpha = \omega$ then $d_x \upharpoonright \omega = x$ ($x \in I$) and $D \upharpoonright \omega = \{d_x \upharpoonright \omega: x \in I\}$. If $\alpha > \omega$ is a limit then we are forced to let

$$d_x \upharpoonright \alpha = \bigcup_{\omega \leq \beta < \alpha} d_x \upharpoonright \beta \quad \text{for } x \in I.$$

One readily checks that (1), (2), and (3) hold in this case.

Now we consider the successor step $\alpha \rightarrow \alpha + 1$. We have to define $d_x(\alpha)$ for $x \in I$. Condition (1) tells us that the map $\phi: x \rightarrow d_x(\alpha)$ is to be a homomorphism from I to 2. Condition (2) tells us that we must have $\phi(x_\alpha) = 1$. For condition (3) we need the following:

$$\begin{aligned} &\text{if } \beta \in [\omega, \alpha], \ F \in [\alpha + 1]^{<\omega}, \text{ and } i \in 2 \\ &\text{then } \{x \in E_\beta: d_x \upharpoonright F = B_\beta^i\} \text{ is infinite.} \end{aligned}$$

If $\alpha \notin F$ this is satisfied because $B_\beta^i \upharpoonright \alpha$ is an accumulation point of $\{d_x \upharpoonright \alpha: x \in E_\beta\}$. So let $F \in [\alpha]^{<\omega}$ and consider $F \cup \{\alpha\}$. Then $d_x \upharpoonright (F \cup \{\alpha\}) = B_\beta^i \upharpoonright (F \cup \{\alpha\})$ means

$$d_x \upharpoonright F = B_\beta^i \upharpoonright F \quad \text{and} \quad \phi(x) = B_\beta^i(\alpha).$$

So what we need is

$$\begin{aligned} &\text{if } \beta \in [\omega, \alpha], \ F \in [\alpha]^{<\omega}, \text{ and } i \in 2 \\ (\oplus) \quad &\text{then } \{x \in E_\beta: d_x \upharpoonright F = B_\beta^i \upharpoonright F \text{ and } \phi(x) = B_\beta^i(\alpha)\} \text{ is infinite.} \end{aligned}$$

This we achieve by applying **MA** to the countable poset

$$\mathbf{P} = \{p \in \text{Fn}(I, 2) : \text{dom}(p) \text{ is a finite subgroup of } I,$$

$$p \text{ is a homomorphism, } x_\alpha \in \text{dom}(p) \text{ and } p(x_\alpha) = 1\},$$

ordered by reverse inclusion. For every $x \in I$, $A_x = \{p: x \in \text{dom}(p)\}$ is dense in $\mathbf{P}$: if $x \notin \text{dom}(p)$ then we can make $p' \leq p$ with $\text{dom}(p') = \langle \langle \text{dom}(p) \cup \{x\} \rangle \rangle$ and $p'(x) = 0$. Thus, if $\mathcal{G}$ is a filter on $\mathbf{P}$ intersecting every A_x then $\phi = \bigcup \mathcal{G}$ is a homomorphism on I satisfying $\phi(x_\alpha) = 1$. For $(\oplus)$ we need more dense sets: for $\beta \in [\omega, \alpha]$, $F \in [\alpha]^{<\omega}$, $i \in 2$, and $n \in \omega$ put

$$\begin{aligned} A(\beta, F, i, n) &= \{p: |\{x \in \text{dom}(p): x \in E_\beta, \\ &\quad d_x \upharpoonright F = B_\beta^i \upharpoonright F; p(x) = B_\beta^i(\alpha)\}| \geq n\}. \end{aligned}$$

Note that $A(\alpha, F, i, 0) = \mathbf{P}$ and that if $p \in A(\beta, F, i, n)$ then there is a $p' \leq p$ with $p' \in A(\beta, F, i, n+1)$. For let $p \in A(\beta, F, i, n)$. Since $\{x \in E_\beta: d_x \upharpoonright F = B_\beta^i \upharpoonright F\}$ is infinite we can pick x from this set with $x \notin \text{dom}(p)$. Then

we can make $p' \leq p$ with $\text{dom}(p') = \langle \langle \text{dom}(p) \cup \{x\} \rangle \rangle$ and $p'(x) = B_p^i(\alpha)$. But then $p' \in A(\beta, F, i, n+1)$. It follows that every $A(\beta, F, i, n+1)$ is dense. We have only $|\alpha| < \mathfrak{c}$ such dense sets so by $\mathbf{MA}_{\text{countable}}$ we can take a filter $\mathcal{G}$ on $\mathbf{P}$ intersecting every A_x and every $A(\beta, i, F, n)$. Then $\phi = \bigcup \mathcal{G}$ is a homomorphism and it satisfies $(\oplus)$. This completes the construction of D and also of G_0 and G_1 and hence of H_0 and H_1 .

3. CONSTRUCTION II

In this section we construct a single countably compact group H such that H^2 is not countably compact. The construction is basically the same as the one in §2 but with more notation. We shall describe what H will look like and then concentrate on the actual construction, trusting that the reader who made it this far will have no problem seeing what is going on.

3.0. A picture of H . For any set of ordinals x we put $\bar{x} = x \times 2$, and we let $\rho: \bar{\tau} \rightarrow \bar{\tau}$ be defined by $\rho(\alpha, i) = \langle \alpha, 1-i \rangle$. Then ρ induces an automorphism (also denoted by ρ) of ${}^{\bar{\tau}}2$. We mentioned that H can be considered to be a subgroup of ${}^{\bar{\tau}}2 \times {}^{\bar{\tau}}2$ containing $H_0 \times \{0\}$ and $\{0\} \times H_1$ where H_0 and H_1 are much like the H_0 and H_1 from §2.

The notation becomes a bit more manageable if we work in ${}^{\bar{\tau}}2$ rather than ${}^{\bar{\tau}}2 \times {}^{\bar{\tau}}2$. For $i \in 2$ put

$$K_i = \{x \in {}^{\bar{\tau}}2: x^{-(1)} \subseteq \mathfrak{c} \times \{i\}\}.$$

Now H will be $D + G_0 + G_1 + G_2$ with D countable and G_0, G_1 , and G_2 ω -bounded. We will let $D_i = D \cap K_i$ for $i \in 2$, and we will have $D = D_0 + D_1$ and $D_1 = \rho(D_0)$. Furthermore we will have $G_i \subseteq K_i$ for $i \in 2$. Then for $i \in 2$ every infinite $E \subseteq D_i$ will have a limit point in G_i so that $H_i = D_i + G_i$ will be countably compact. We will also have $G_1 \cap \rho(G_0) = \{0\}$ and $D \cap (G_0 + G_1) = \{0\}$, and this will imply that $H_0 \times H_1$ is not countably compact. We will use G_2 to provide accumulation points for the infinite subsets of $D \setminus (D_0 \cup D_1)$.

3.1. The construction.

3.1.0. Let $\mathcal{A}$ be the ω -independent family on $\mathfrak{c}$ from 1.1. We split $\mathcal{A}$ into four disjoint pieces $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ of size $\mathfrak{c}$. We define $\mathcal{A}_0^+ = \{A \times \{0\}: A \in \mathcal{A}_0\}$, $\mathcal{A}_1^+ = \{A \times \{1\}: A \in \mathcal{A}_1\}$, and $\mathcal{A}_i^+ = \{A \times 2: A \in \mathcal{A}_i\}$ ($i = 2, 3$). As before we identify every $A \in \mathcal{A}_0^+ \cup \mathcal{A}_1^+ \cup \mathcal{A}_2^+ \cup \mathcal{A}_3^+$ with its characteristic function in ${}^{\bar{\tau}}2$.

3.1.1. Let $I = \{x \in {}^{\bar{\omega}}2: x^{-(1)} \text{ is finite}\}$ and for $i \in 2$ let $I_i = \{x \in I: x^{-(1)} \subseteq \omega \times \{i\}\}$. Then $I = I_0 + I_1$ and each $x \in I$ is written (uniquely) as $x_0 + x_1$ with $x_0 \in I_0$ and $x_1 \in I_1$. We will have $D = \{d_x: x \in I\}$ and $D_i = \{d_x: x \in I_i\}$ ($i \in 2$).

3.1.2. To see for what infinite sets $E \subseteq D$ we have to provide accumulation points let $E \subseteq I$ be infinite. We can have (i) $E \cap I_0$ is infinite, but $\{d_x: x \in E \cap I_0\}$ will have an accumulation point in G_0 ; (ii) $E \cap I_1$ is infinite, now G_1

provides an accumulation point; (iii) for infinitely many $x \in E$, x is symmetric (i.e., $x = \rho(x)$), in this case we will provide an accumulation point in G_2 . If none of the above cases occur then without loss of generality for all $x \in E$, $x_0, x_1, x + \rho(x) \neq \underline{0}$. In this case we can have an infinite $E' \subseteq E$ and an $a \in I$ such that (iv) $\forall x \in E', x_0 = a$, (v) $\forall x \in E', x_1 = a$ or (vi) $\forall x \in E', x + \rho(x) = a$.

In cases (iv) or (v) $d_a + g$ will be an accumulation point of $\{d_x: x \in E'\}$ when g is an accumulation point of $\{d_x + d_a: x \in E'\}$ (from (i) or (ii)). In case (v) take $E'' \subseteq E'$ infinite and $b \in I$ such that $\forall x \in E'', x \cap (x + \rho(x)) = b$. Then $x = b + (x \cap \rho(x))$ for $x \in E''$ and $d_b + g$ is an accumulation point of $\{d_x: x \in E''\}$ (from (iii)).

Finally if (iv), (v), and (vi) do not occur either, we get an infinite $E' \subseteq E$ such that

$$(*) \quad \forall x, y \in E': x \neq y \rightarrow x_0 \neq y_0, x_1 \neq y_1, \text{ and } x + \rho(x) \neq y + \rho(y).$$

We conclude that it suffices to provide accumulation points for $\{d_x: x \in E\}$ in case $E \subseteq I_0$, $E \subseteq I_1$, every $x \in E$ is symmetric, or E satisfies $(*)$.

3.1.3. We enumerate everything in sight [Ru].

- $\langle A_\alpha^i: \omega \leq \alpha < c \rangle$ enumerates $\mathcal{A}_i^+$ without repetitions ($i \in 4$),
- $\langle E_\alpha^i: \omega \leq \alpha < c \rangle$ enumerates $[I_i]^\omega$ ($i \in 2$),
- $\langle E_\alpha^2: \omega \leq \alpha < c \rangle$ enumerates $\{E \in [I]^\omega: E \text{ is symmetric}\}$,
- $\langle E_\alpha^3: \omega \leq \alpha < c \rangle$ enumerates $\{E \in [I]^\omega: E \text{ satisfies } (*)\}$, and
- $\langle \langle x_\alpha, \mathcal{E}_\alpha \rangle: \omega \leq \alpha < c \rangle$ enumerates $(I \setminus \{\underline{0}\}) \times [\mathcal{A}]^\omega$ in such a way that for every $\langle x, \mathcal{E} \rangle$ the set $\{\alpha: \langle x_\alpha, \mathcal{E}_\alpha \rangle = \langle x, \mathcal{E} \rangle \text{ and } \alpha \notin \bigcup \mathcal{E}\}$ has cardinality c .

3.1.4. By induction on $\alpha \in [\omega, c]$ we construct $D \restriction \bar{\alpha} = \{d_x \restriction \bar{\alpha}: x \in I\}$ satisfying:

- (0) $\forall x \in I: d_x \restriction \bar{\omega} = x$ and if $\omega \leq \beta < \alpha$ then $d_x \restriction \bar{\beta} \subseteq d_x \restriction \bar{\alpha}$,
- (1) $\forall x \in I: d_{\rho(x)} \restriction \bar{\alpha} = \rho(d_x \restriction \bar{\alpha})$ and $\forall x, y \in I: d_x \restriction \bar{\alpha} + d_y \restriction \bar{\alpha} = d_{x+y} \restriction \bar{\alpha}$,
- (2) $\forall x \in I_i: d_x \restriction (1) \cap \bar{\alpha} \subseteq \alpha \times \{i\}$ ($i \in 2$), and
- (3) if $\omega \leq \beta < \alpha$ then $d_{x_\beta}(\beta, 0) = 1$ if $x_\beta \notin I_1$, and $d_{x_\beta}(\beta, 1) = 1$ if $x_\beta \notin I_0$.

Once $D \restriction \bar{\alpha}$ is found we pick for $i = 0, 1, 2, 3$ an accumulation point x_α^i for $\{d_x \restriction \bar{\alpha}: x \in E_\alpha^i\}$ in $\bar{\alpha}^2$. Then for $i \in 4$ we put $B_\alpha^i = x_\alpha^i \cup A_\alpha^i \restriction [\bar{\alpha}, c)$; note that $B_\alpha^i \in K_i$ for $i \in 2$. We require for $i = 0, 1, 2, 3$:

- (4) if $\omega \leq \beta < \alpha$ then $B_\beta^i \restriction \bar{\alpha}$ is an accumulation point of $\{d_x \restriction \bar{\alpha}: x \in E_\beta^i\}$.

3.1.5. If $\alpha = \omega$ we put $d_x \restriction \bar{\omega} = x$ ($x \in I$) and $D \restriction \bar{\omega} = I$. If α is a limit we put

$$d_x \restriction \bar{\alpha} = \bigcup_{\omega \leq \beta < \alpha} d_x \restriction \bar{\beta} \quad (x \in I)$$

and an easy check gives (0)–(4).

3.1.6. Now consider the successor step $\alpha \rightarrow \alpha + 1$. As in §2 we use a homomorphism ϕ to define $d_x(\alpha, 0)$ and $d_x(\alpha, 1)$ for all $x \in I$.

Now we will have $\phi: I \rightarrow {}^2 2$. We need

$$\begin{aligned} \phi(\rho(x))(0) &= \phi(x)(1) \quad \text{and} \quad \phi(\rho(x))(1) = \phi(x)(0) \quad \text{for (1),} \\ \phi(x)(1) &= 0 \quad (x \in I_0) \quad \text{and} \quad \phi(x)(0) = 0 \quad (x \in I_1) \quad \text{for (2),} \\ \phi(x_\alpha)(0) &= 1 \quad \text{if } x_\alpha \notin I_1 \quad \text{and} \quad \phi(x_\alpha)(1) = 1 \quad \text{if } x_\alpha \notin I_0 \quad \text{for (3).} \end{aligned}$$

We use the countable poset

$$\begin{aligned} \mathbf{P} = \{ & \rho \in \text{Fn}(I, {}^2 2) : \text{dom}(\rho) \text{ is a finite subgroup of } I; \rho \text{ is a homomorphism;} \\ & \text{if } x \in \text{dom}(\rho) \text{ then } x_0, x_1, \rho(x) \in \text{dom}(\rho) \text{ and } \rho(x_0)(1) = \rho(x_1)(0) \\ & = 0 \text{ and } \rho(\phi(x))(i) = \rho(x)(1 - i) \quad (i \in 2); x_\alpha \in \text{dom}(\rho) \text{ and} \\ & x_\alpha \notin I_0 \rightarrow \rho(x_\alpha)(1) = 1, \quad x_\alpha \notin I_1 \rightarrow \rho(x_\alpha)(0) = 1 \}, \end{aligned}$$

ordered by reverse inclusion.

Let $x \in I$ and let $\mathcal{A}_x = \{p: x \in \text{dom}(p)\}$. To show $\mathcal{A}_x$ is dense consider the group F_x generated by $\{x_0, x_1, \rho(x_0), \rho(x_1)\}$. We have to define $p' \leq p$ with $\text{dom}(p') = \text{dom}(p) + F_x$. The special demands on $\text{dom}(p)$ lead to the following four possibilities for $F_x \cap \text{dom}(p)$:

- $\text{dom}(p) \cap F_x = \{\underline{0}\}$,
in this case $p'(x_i)(0) = p'(x_i)(1) = 0 \quad (i \in 2)$ works;
- $\text{dom}(p) \cap F_x = \{\underline{0}, x_0, \rho(x_0), x_1 + \rho(x_1)\}$,
in this case $p'(x_1)(0) = p'(x_1)(1) = 0$ works;
- $\text{dom}(p) \cap F_x = \{\underline{0}, x_1, \rho(x_1), x_1 + \rho(x_1)\}$,
in this case $p'(x_0)(0) = p'(x_0)(1) = 0$ works;
- $\text{dom}(p) \cap F_x = \{\underline{0}, x_0 + \rho(x_1), x_1 + \rho(x_0), x + \rho(x)\}$,
in this case $\varepsilon = p(x + \rho(x))$ is symmetric: $\varepsilon(0) = \varepsilon(1) = i$. Now let $p'(x_0)(0) = i, p'(x_0)(1) = 0$ and $p'(x_1)(1) = 0$; this works.

Thus if $\mathcal{G}$ is a filter on $\mathbf{P}$ intersecting all $\mathcal{A}_x$ then $\phi = \bigcup \mathcal{G}$ is a homomorphism taking care of everything except possibly (4). For (4) we consider the following sets: for $\beta \in [\omega_0, \alpha]$, $F \in [\bar{\alpha}]^{<\omega}$, $i \in 4$, and $n \in \omega$ let

$$\begin{aligned} A(\beta, F, i, n) = \{ & p: |x \in \text{dom}(p): x \in E_\beta, d_x \upharpoonright F = B_\beta^i \upharpoonright F, \\ & (x)(j) = B_\beta^i(\alpha, j) \quad (j \in 2)| \geq n \}. \end{aligned}$$

Then $A(\beta, F, i, 0) = \mathbf{P}$ and if $p \in A(\beta, F, i, n)$ then there is a $p' \leq p$ with $p' \in A(\beta, F, i, n+1)$:

$i = 0, 1$: Pick $x \in E_\beta^i$ with $d_x \upharpoonright F = B_\beta^i \upharpoonright F$ and $x \notin \text{dom}(p)$. Then $F_x = \{\underline{0}, x, \rho(x), x + \rho(x)\}$ and $F_x \cap \text{dom}(p) = \{\underline{0}\}$. Define $p' \leq p$ by $\text{dom}(p') = \text{dom}(p) + F_x$ and $p'(x)(j) = B_\beta^i(\alpha, j) \quad (j \in 2)$. Note that $B_\beta^i(\alpha, 1 - i) = 0$, so that this definition is in order.

$i = 2$: Again pick $x \in E_\beta^2$ with $d_x \upharpoonright F = B_\beta^i \upharpoonright F$ and $x \notin \text{dom}(p)$. Then $F_x = \{0, x_0, x_1, x\}$ and $F_x \cap \text{dom}(p) = \{0\}$. Note that $B_\beta^1(\alpha, 0) = B_\beta^2(\alpha, 1)$, since $\alpha \geq \beta$, so that the following safely defines $p' \leq p$ with $\text{dom}(p') = \text{dom}(p) + F_x$: $p'(x_0)(0) = p'(x_1)(1) = B_\beta^2(\alpha, 0)$ and $p'(x_0)(1) = p'(x_1)(0) = 0$.

$i = 3$: Now we can pick $x \in E_\beta^3$ with $d_x \upharpoonright F = B_\beta^3 \upharpoonright F$ and $x_0, x_1, x_0 + \rho(x_1) \notin \text{dom}(p)$. Again $F_x \cap \text{dom}(p) = \{0\}$, and since $B_\beta^3(\alpha, 0) = B_\beta^3(\alpha, 1)$ we can (again) safely define $p' \leq p$ by $\text{dom}(p') = \text{dom}(p) + F_x$, and $p'(x_0)(0) = p'(x_1)(1) = B_\beta^3(\alpha, 0)$ and $p'(x_0)(1) = p'(x_1)(0) = 0$.

In the end we have $d_x = d_x \upharpoonright \bar{\tau}$ ($x \in I$) and $D = D \upharpoonright \bar{\tau}$. We let G_0 be the ω -bounded subgroup of ${}^{\bar{\tau}}2$ generated by $\mathcal{B}_0$ and similarly $\mathcal{B}_1$ generates G_1 and $\mathcal{B}_2 \cup \mathcal{B}_3$ generates G_2 .

3.1.7. We put $H_0 = D_0 + G_0$, $H_1 = D_1 + G_1$, and $H = D + G_0 + G_1 + G_2$. By (3) we know that for every countable $\mathcal{E} \subseteq \mathcal{A}$ and for every $d \in D$,

$$|d^{\leftarrow}(1) \setminus (\bigcup \mathcal{E} \times 2)| = \mathfrak{c}.$$

This readily implies that $|D \cap (G_0 + G_1 + G_2)| = \{0\}$. The considerations given in 3.1.1 now give us, using (4), that every infinite subset of D has an accumulation point in H . Together with the ω -boundedness of $G_0 + G_1 + G_2$ this yields that H is countably compact. Next because $G_0 \cap \rho(G_1) = \{0\}$ we see that $\Delta = \{\langle d, \rho(d) \rangle : d \in D_0\}$ is closed in $H_0 \times H_1$ whence $H_0 \times H_1$ is not countably compact. Finally $H_0 = H \cap K_0$ and $H_1 = H \cap K_1$ so that $H_0 \times H_1$ is closed in H^2 so that H^2 is not countably compact. (The equalities $G_0 \cap \rho(G_1) = \{0\}$, $H_0 = H \cap K_0$, and $H_1 = H \cap K_1$ follow easily from 1.1(b), just as in 2.2.)

3.2. Remark. It seems that our example needs $\mathbf{MA}_{\text{countable}}$ in an essential way for the same reasons as mentioned in [vD]. $\mathbf{MA}_{\text{countable}}$ implies that every $u \in \omega^*$ has character $\mathfrak{c}$. Assume $2^\omega = 2^{\omega_1} = \mathfrak{c}$. Then we can choose an ω_1 -independent family on $\mathfrak{c}$ and make the group $G_0 + G_1 + G_2$ ω_1 -bounded (these concepts have the obvious meanings). Then H will be initially ω_1 -compact (meaning that every open cover of size $\leq \omega_1$ has a finite subcover). But this variation does not work without $\mathbf{MA}_{\text{countable}}$: if there is an ultrafilter of character ω_1 on ω then the product of any family of initially ω_1 -compact spaces is countably compact.

REFERENCES

- [CR] W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math. **16** (1966), 483–496.
- [vD] E. K. van Douwen, *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (1980), 417–427.

- [Ru] M. E. Rudin, *Interview*, College Math. J. **19** (1988), 114–137.
- [We] W. A. R. Weiss, *Versions of Martin's Axiom*, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 827–886.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, TU DELFT, JULIANALAAN 132, 2628 BL DELFT, THE NETHERLANDS

E-mail address: WIAWKPH @ DUTRUN. TUDELFT.NL or WIAWKPH @ HDETUD1. BITNET

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, VRIJE UNIVERSITEIT, DE BOELELAAN 1081, 1081 HV AMSTERDAM, THE NETHERLANDS

E-mail address: VANMILL @ CS. VU. NL